# Supplement to "What Model for Entry in First-Price Auctions? A Nonparametric Approach"

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## S.1 Introduction and notation

This paper contains supplemental materials for Marmer, Shneyerov, and Xu (2011), MSX hereafter. It establishes validity of the bootstrap delta-method expansion (65) in Appendix E of MSX.

In what follows, the statistics with superscript $^\dagger$ denote the bootstrap analogues of the statistics computed using the original data. To simplify the notion, we will suppress the subscript indicating the bootstrap sample number for bootstrap objects $(m)$. Let $P^\dagger$ denote probability conditional on the original sample. We use $E^\dagger$ and $Var^\dagger$ to denote expectation and variance under $P^\dagger$ respectively.

Let $\pi^\dagger$ denote the distribution of $N_l^\dagger$ implied by $P^\dagger$, i.e.

$$\begin{aligned}\pi^\dagger(N) &= P^\dagger(N_l^\dagger = N)\\ &= L^{-1}\sum_{l=1}^{L} 1(N_l = N)\\ &= \hat{\pi}\,(N)\,,\end{aligned}$$

where $\pi(N) = P(N_l = N)$. Also, define

$$\begin{aligned} p^\dagger(N) &= P^\dagger(y_{il}^\dagger = 1|N) \\ &= \sum_{l=1}^{L} (n_l/N)\, P^\dagger\left(n_l^\dagger = n_l|N\right) \\ &= \frac{\sum_{l=1}^{L} (n_l/N)\, 1\{N_l = N\}}{\sum_{l=1}^{L} 1\{N_l = N\}} \\ &= \hat{p}\,(N)\,, \end{aligned}$$

where $p\,(N) = P\,(y_{il} = 1|N_l = N)$.

We say $\zeta_L = O_p^\dagger(\lambda_L)$ if for all $\varepsilon > 0$ there is $\Delta_\varepsilon > 0$ such that for all $L \geq L_\varepsilon$, $P(P^\dagger(|\zeta_L/\lambda_L| > \Delta_\varepsilon) > \varepsilon) < \varepsilon$. We say $\zeta_L = o_p^\dagger(\lambda_L)$ if $P^\dagger(|\zeta_L/\lambda_L| > \varepsilon) \to_p 0$ for all $\varepsilon > 0$ as $L \to \infty$.

## S.2 Auxiliary results

In this section, we present some simple results concerning the stochastic order (with respect to $P^\dagger$) of the bootstrap statistics. Let $\hat{\theta}_L$ be a statistic computed using the data in the original sample, and let $\hat{\theta}_L^\dagger$ be the bootstrap analogue of $\hat{\theta}_L$.

**Lemma S.1** **(a)** *Suppose that* $\hat{\theta}_L = \theta + o_p(\delta_L)$ *and* $\hat{\theta}_L^\dagger = \hat{\theta}_L + o_p^\dagger(\delta_L)$. *Then,* $\hat{\theta}_L^\dagger = \theta + o_p^\dagger(\delta_L)$.

**(b)** *Suppose that* $\hat{\theta}_L = \theta + O_p(\delta_L)$ *and* $\hat{\theta}_L^\dagger = \hat{\theta}_L + O_p^\dagger(\delta_L)$. *Then,* $\hat{\theta}_L^\dagger = \theta + O_p^\dagger(\delta_L)$.

**Proof.** For part (a), since $\hat{\theta}_L$ is not random under $P^\dagger$,

$$\begin{aligned} P^\dagger\left(\delta_L^{-1}\left|\hat{\theta}_L^\dagger - \theta\right| > \varepsilon\right) &\leq P^\dagger\left(\delta_L^{-1}\left|\hat{\theta}_L - \theta\right| > \frac{\varepsilon}{2}\right) + P^\dagger\left(\delta_L^{-1}\left|\hat{\theta}_L^\dagger - \hat{\theta}_L\right| > \frac{\varepsilon}{2}\right) \\ &= 1\left(\delta_L^{-1}\left|\hat{\theta}_L - \theta\right| > \frac{\varepsilon}{2}\right) + o_p(1)\,. \end{aligned}$$

For the first summand, we have that for all $\varepsilon, \eta > 0$,

$$P\left(1\left(\delta_L^{-1}\left|\hat{\theta}_L - \theta\right| > \frac{\varepsilon}{2}\right) > \eta\right) = P\left(\delta_L^{-1}\left|\hat{\theta}_L - \theta\right| > \frac{\varepsilon}{2}\right) \to 0.$$

The proof of part (b) is similar. ■

**Lemma S.2** *Suppose that* $E^{\dagger}(\hat{\theta}_L^{\dagger})^2 = O_p(\lambda_L^2)$. *Then* $\hat{\theta}_L^{\dagger} = O_p^{\dagger}(\lambda_L)$.

**Proof.** Since $E^{\dagger}(\hat{\theta}_L^{\dagger})^2 = O_p(\lambda_L^2)$, for all $\varepsilon > 0$ there is $\Delta_\varepsilon > 0$ such that $P(E^{\dagger}(\hat{\theta}_L^{\dagger})^2 > \Delta_\varepsilon^2 \lambda_L^2) < \varepsilon$. Let $\tilde{\Delta}_\varepsilon^2 = \Delta_\varepsilon^2/\varepsilon$. Then, we can write

$$P(E^{\dagger}(\hat{\theta}_L^{\dagger})^2 > \tilde{\Delta}_\varepsilon^2 \varepsilon \lambda_L^2) < \varepsilon \tag{S.1}$$

for all $L$ large enough. By Markov's inequality,

$$P^{\dagger}\left(\left|\frac{\hat{\theta}_L^{\dagger}}{\lambda_L}\right| \geq \tilde{\Delta}_\varepsilon\right) \leq \frac{E^{\dagger}\left(\hat{\theta}_L^{\dagger}\right)^2}{\lambda_L^2 \tilde{\Delta}_\varepsilon^2}.$$

Thus, for all $\varepsilon > 0$ there is $\tilde{\Delta}_\varepsilon$, such that for all $L$ large enough,

$$P\left(P^{\dagger}\left(\left|\frac{\hat{\theta}_L^{\dagger}}{\lambda_L}\right| \geq \tilde{\Delta}_\varepsilon\right) > \varepsilon\right) \leq P\left(\frac{E^{\dagger}\left(\hat{\theta}_L^{\dagger}\right)^2}{\tilde{\Delta}_\varepsilon^2 \lambda_L^2} > \varepsilon\right) < \varepsilon,$$

where the last inequality is by (S.1). ■

## S.3 Main result

The validity of (65) in MSX follows from Lemma S.3 below, which is similar to Lemma 3 in MSX. Given the results in Lemma S.3, (65) can be shown by the same arguments as those in the proof of Proposition 7 in Appendix C in MSX, and by applying Lemma S.1.

**Lemma S.3** *Suppose that assumptions of Lemma 3 in MSX hold. Then, for all* $x$ *in the interior of* $\mathcal{X}$ *and* $N \in \mathcal{N}$,

**(a)** $\hat{\varphi}^{\dagger}(x) = \hat{\varphi}(x) + O_p^{\dagger}\left(Lh^d\right)^{-1/2}$.

**(b)** $\hat{\pi}^{\dagger}(N|x) = \hat{\pi}(N|x) + O_p^{\dagger}\left(Lh^d\right)^{-1/2}$.

**(c)** $\hat{p}^{\dagger}(N,x) = \hat{p}(N,x) + O_p^{\dagger}((Lh^d/\log L)^{-1/2} + h^R)$.

**(d)** $\sup_{b \in [\underline{b}(N,x), \bar{b}(N,x)]} |\hat{G}^{*,\dagger}(b|N,x) - \hat{G}^{*}(b|N,x)| = O_p^{\dagger}((Lh^d/\log L)^{-1/2} + h^R)$.

**(e)** $\sup_{\tau\in[\varepsilon,1-\varepsilon]}|\hat{q}^{*,\dagger}(\tau|N,x)-\hat{q}^{*}(\tau|N,x)|=O_p^{\dagger}((Lh^d/\log L)^{-1/2}+h^R)$, *for any* $0<\varepsilon<1/2$.

**(f)** $\sup_{b\in[b_1(N,x),b_2(N,x)]}|\hat{g}^{*,\dagger}(b|N,x)-\hat{g}^{*}(b|N,x)|=O_p^{\dagger}((Lh^{d+1}/\log L)^{-1/2}+h^R)$, *where* $b_1(N,x)$ *and* $b_2(N,x)$ *are defined in (40) and (41) in MSX.*

**(g)** $\sup_{\tau\in[\tau_1(N,x),\tau_2(N,x)]}|\hat{Q}^{*,\dagger}(\tau|N,x)-\hat{Q}^{*}(\tau|N,x)|=O_p^{\dagger}((Lh^{d+1}/\log L)^{-1/2}+h^R)$.

**(h)** $\hat{Q}^{*,\dagger}(\hat{\beta}^{\dagger}(\tau,N,x)|N,x)=\hat{Q}^{*}(\beta(\tau,N,x)|N,x)+O_p^{\dagger}((Lh^{d+1}/\log L)^{-1/2}+h^R)$ *uniformly in* $\tau$ *such that* $\beta(\tau,N,x)\in[\tau_1(N,x)+\varepsilon,\tau_2(N,x)-\varepsilon]$, *for any* $0<\varepsilon<(\tau_2(N,x)-\tau_1(N,x))/2$.

**Proof.** Part (a) follows from the uniform strong approximation in Chen and Lo (1997), Proposition 3.2.

For part (b), write

$$\hat{\pi}(N|x)=\hat{\pi}(N,x)\hat{\varphi}(x)\text{, where}$$
$$\hat{\pi}(N,x)=\frac{1}{Lh^d}\sum_{l=1}^{L}1(N_l=N)\textstyle\prod_{k=1}^{d}K\left(\frac{x_{kl}-x_k}{h}\right).$$

By Proposition 3.2 in Chen and Lo (1997), $(Lh^d)^{1/2}(\hat{\pi}(N,x)-E\hat{\pi}(N,x))=O_p^{\dagger}(1)$. By the Taylor expansion of $\hat{\pi}^{\dagger}(N|x)$, the result in part (a), and since $\hat{\varphi}(x)$ is bounded away from zero with probability approaching one by Assumption 3(b) and Lemma 3(a) in MSX,

$$\begin{aligned}\left(Lh^d\right)^{1/2}\left(\hat{\pi}^{\dagger}(N|x)-\hat{\pi}(N|x)\right)=&\frac{1}{\hat{\varphi}(x)}\left(Lh^d\right)^{1/2}\left(\hat{\pi}^{\dagger}(N,x)-\hat{\pi}(N,x)\right)\\&-\frac{\hat{\pi}^{\dagger}(N,x)}{(\hat{\varphi}(x))^2}\left(Lh^d\right)^{1/2}\left(\hat{\varphi}^{\dagger}(x)-\hat{\varphi}(x)\right)\\&+o\left(\left(Lh^d\right)^{1/2}\left(\hat{\varphi}^{\dagger}(x)-\hat{\varphi}(x)\right)\right)\\=&O_p^{\dagger}(1).\end{aligned}$$

The proof of part (c) is similar to that of part (b) and therefore omitted.

We prove part (d) next. The proof is similar to the proof of Lemma B.1 in Newey (1994). For fixed $x$ in the interior of $\mathcal{X}$ and $N\in\mathcal{N}$, write

$$\hat{G}^{*}(b^{*},N,x)=\hat{G}(b^{*}|N,x)\hat{p}(N,x)\hat{\pi}(N|x)\hat{\varphi}(x),$$

so that

$$\hat{G}^{*}(b,N,x)=\frac{1}{NL}\sum_{l=1}^{L}\sum_{i=1}^{n_l}T_{il},$$
$$T_{il}=\frac{1}{h^d}y_{il}1(b_{il}\leq b)1\{N_l=N\}\prod_{k=1}^{d}K\left(\frac{x_{kl}-x_k}{h}\right), \tag{S.2}$$

and let

$$\hat{G}^{*,\dagger}(b,N,x)=\frac{1}{nL}\sum_{l=1}^{L}\sum_{i=1}^{n_l}T_{il}^{\dagger}(b),$$
$$T_{il}^{\dagger}(b)=\frac{1}{h^d}y_{il}^{\dagger}1\left(b_{il}^{\dagger}\leq b\right)1\left\{N_l^{\dagger}=N\right\}\prod_{k=1}^{d}K\left(\frac{x_{kl}^{\dagger}-x_k}{h}\right).$$

Next, for the chosen values $N$ and $x$, let

$$I=\left[\underline{b}(N,x),\bar{b}(N,x)\right],$$
$$I=\cup_{k=1}^{J_L}I_k,$$

where the sub-intervals $I_k$'s are non-overlapping and of length

$$s_L=\frac{\log L}{L}. \tag{S.3}$$

Denote as $c_k$ the center of $I_k$. Note that $I, I_k, c_k$ depend on $N$ and $x$. Denote as $\kappa(b)$ the interval containing $b$, i.e. $b\in I_{\kappa(b)}$. Since

$$\hat{G}^{*}(b,N,x)=E^{\dagger}T_{il}^{\dagger}(b),$$

we can write

$$\hat{G}^{*,\dagger}(b,n,x)-\hat{G}^{*}(b,n,x)=A_L^{\dagger}(b)-B_L^{\dagger}(b)+C_L^{\dagger}(b),\text{ where}$$
$$A_L^{\dagger}(b)=\frac{1}{NL}\sum_{l=1}^{L}\sum_{i=1}^{N_l}\left(T_{il}^{\dagger}(b)-T_{il}^{\dagger}\left(c_{\kappa(b)}\right)\right),$$
$$B_L^{\dagger}(b)=\frac{1}{NL}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\left(E^{\dagger}T_{il}^{\dagger}(b)-E^{\dagger}T_{il}^{\dagger}\left(c_{\kappa(b)}\right)\right),$$

$$C_L^\dagger(b) = \frac{1}{NL}\sum_{l=1}^{L}\sum_{i=1}^{n_l}\left(T_{il}^\dagger\left(c_{\kappa(b)}\right) - E^\dagger T_{il}^\dagger\left(c_{\kappa(b)}\right)\right).$$

In the above decomposition, $A_L^\dagger(b)$ is the average of the deviations of $T_{il}^\dagger(b)$ from its value computed using the center of the interval containing $b$, and $B_L^\dagger(b)$ is the expected value under $P^\dagger$ of $A_L^\dagger(b)$. The terms $\sup_{b\in I}|A_L^\dagger(b)|$ and $\sup_{b\in I}|B_L^\dagger(b)|$ are small when $s_L$ is small.

For $A_L^\dagger$ we have

$$\begin{aligned}
&\left|T_{il}^\dagger(b) - T_{il}^\dagger\left(c_{\kappa(b)}\right)\right| \\
&\leq h^{-d}(\sup K)^d y_{il}^\dagger 1\left(N_l^\dagger = N\right)\left|1\left(b_{il}^\dagger \leq b\right) - 1\left(b_{il}^\dagger \leq c_{\kappa(b)}\right)\right| \\
&\leq h^{-d}(\sup K)^d y_{il}^\dagger 1\left(N_l^\dagger = N\right) 1\left(b_{il}^\dagger \in I_{\kappa(b)}\right), \qquad \text{(S.4)}
\end{aligned}$$

where the second inequality holds because $|1(b_{il}^\dagger \leq b) - 1(b_{il}^\dagger \leq c_{\kappa(b)})|$ is equal to zero if $b_{il}^\dagger \notin I_{\kappa(b)}$ and is at most 1 if $b_{il}^\dagger \in I_{\kappa(b)}$. Thus,

$$\left|A_L^\dagger(b)\right| \leq h^{-d}(\sup K)^d \frac{1}{NL}\sum_{l=1}^{L}\sum_{i=1}^{N} y_{il}^\dagger 1\left(N_l^\dagger = N\right) 1\left(b_{il}^\dagger \in I_{\kappa(b)}\right). \qquad \text{(S.5)}$$

Next,

$$\begin{aligned}
&E^\dagger\left(\frac{1}{NL}\sum_{l=1}^{L}\sum_{i=1}^{N} y_{il}^\dagger 1\left(N_l^\dagger = N\right) 1\left(b_{il}^\dagger \in I_k\right)\right) \\
&= E^\dagger\left(y_{il}^\dagger 1\left(N_l^\dagger = N\right) 1\left(b_{il}^\dagger \in I_k\right)\right) \\
&= E^\dagger\left(y_{il}^\dagger 1\left(N_l^\dagger = N\right) P^\dagger\left(b_{il}^\dagger \in I_k | y_{il}^\dagger = 1, N_l^\dagger = N\right)\right) \\
&= E^\dagger\left(1\left(N_l^\dagger = N\right) p^\dagger(N) P^\dagger\left(b_{il}^\dagger \in I_k | y_{il}^\dagger = 1, N_l^\dagger = N\right)\right) \\
&= \pi^\dagger(N) p^\dagger(N) P^\dagger\left(b_{il}^\dagger \in I_k | y_{il}^\dagger = 1, N_l^\dagger = N\right).
\end{aligned}$$

Further,

$$E^\dagger\left[\frac{1}{nL}\sum_{l=1}^{L}\sum_{i=1}^{N} y_{il}^\dagger 1\left(N_l^\dagger = N\right) 1\left(b_{il}^\dagger \in I_k\right)\right.$$

$$- P^{\dagger}\left(b_{il}^{\dagger} \in I_k | y_{il}^{\dagger} = 1, N_l^{\dagger} = N\right) \pi^{\dagger}(N) p^{\dagger}(N)\Big]^2 \leq$$

$$\leq \frac{P^{\dagger}\left(b_{il}^{\dagger} \in I_k | y_{il}^{\dagger} = 1, N_l^{\dagger} = N\right) \pi^{\dagger}(N) p^{\dagger}(N)}{NL}, \tag{S.6}$$

and by Lemma S.2,

$$\begin{aligned}
&\frac{1}{nL} \sum_{l=1}^{L} \sum_{i=1}^{N} y_{il}^{\dagger} 1\left(N_l^{\dagger} = N\right) 1\left(b_{il}^{\dagger} \in I_k\right) \\
&= P^{\dagger}\left(b_{il}^{\dagger} \in I_k | y_{il}^{\dagger} = 1, N_l^{\dagger} = N\right) \pi^{\dagger}(N) p^{\dagger}(N) \\
&\quad + O_p^{\dagger}\left(\frac{P^{\dagger}\left(b_{il}^{\dagger} \in I_k | y_{il}^{\dagger} = 1, N_l^{\dagger} = N\right) \pi^{\dagger}(N) p^{\dagger}(N)}{NL}\right)^{1/2} \\
&= P^{\dagger}\left(b_{il}^{\dagger} \in I_k | y_{il}^{\dagger} = 1, N_l^{\dagger} = N\right) \pi^{\dagger}(N) p^{\dagger}(N) \\
&\quad \times \left(1 + O_p^{\dagger}\left(\frac{1}{P^{\dagger}\left(b_{il}^{\dagger} \in I_k | y_{il}^{\dagger} = 1, N_l^{\dagger} = N\right) \pi^{\dagger}(N) p^{\dagger}(N) NL}\right)^{1/2}\right).
\end{aligned} \tag{S.7}$$

Now, by a similar argument,

$$\begin{aligned}
&P^{\dagger}\left(b_{il}^{\dagger} \in I_k | y_{il}^{\dagger} = 1, N_l^{\dagger} = N\right) \pi^{\dagger}(N) p^{\dagger}(N) \\
&= \frac{1}{NL} \sum_{l=1}^{L} \sum_{i=1}^{N} y_{il} 1(N_l = N) 1(b_{il} \in I_k) \\
&= P(b_{il} \in I_k | y_{il} = 1, N_l = N) \pi(N) p(N) \\
&\quad \times \left(1 + O_p\left(\frac{1}{P(b_{il} \in I_k | y_{il} = 1, N_l = N) \pi(N) p(N) NL}\right)^{1/2}\right) \\
&\leq \sup_{k=1,\ldots,J_L} P(b_{il} \in I_k | y_{il} = 1, N_l = N) \pi(N) p(N) \\
&\quad \times \left(1 + O_p\left(\frac{1}{\inf_{k=1,\ldots,J_L} P(b_{il} \in I_k | y_{il} = 1, N_l = N) \pi(N) p(N) NL}\right)^{1/2}\right).
\end{aligned} \tag{S.8}$$

Furthermore, for all $I_k$'s

$$\left(\inf_{b\in I,x\in\mathcal{X}} g^*\left(b|N,x\right)\right)s_L \leq P\left(b_{il}\in I_k|y_{il}=1,N_l=N\right)\leq\left(\sup_{b\in I,x\in\mathcal{X}} g^*\left(b|N,x\right)\right)s_L. \tag{S.9}$$

Equations (S.5)-(S.9) together imply that

$$\begin{aligned}\left|\sup_{b\in I} A_L^\dagger\left(b\right)\right| &= O_p^\dagger\left(h^{-d}s_L\left(1+O_p\left(\left(\frac{1}{s_L L}\right)^{1/2}\right)\right)\right)\\ &= O_p^\dagger\left(\frac{\log L}{Lh^d}\right),\end{aligned} \tag{S.10}$$

where the last equality is by (S.3).

By (S.4), (S.8), and (S.9), for $B_L^\dagger\left(b\right)$ we have

$$\begin{aligned}\left|\sup_{b\in I} B_L^\dagger\left(b\right)\right| &\leq \sup_{b\in I} E^\dagger\left|T_{il}^\dagger\left(b\right)-T_{il}^\dagger\left(c_{\kappa(b)}\right)\right|\\ &\leq h^{-d}\left(\sup K\right)^d\pi^\dagger\left(n\right)\sup_{k=1,\ldots,J_L} P^\dagger\left(b_{il}^\dagger\in I_k|y_{il}^\dagger=1,N_l^\dagger=N\right)\\ &= O_p^\dagger\left(\frac{\log L}{Lh^d}\right).\end{aligned} \tag{S.11}$$

Note that $C_L^\dagger(b)$ depends on $b$ only through $c_k$'s, and therefore

$$\sup_{b\in I}|C_L^\dagger(b)|\leq\max_{k=1,\ldots,J_L}|C_L^\dagger(c_k)|. \tag{S.12}$$

A Bonferroni inequality implies that for any $\Delta>0$,

$$\begin{aligned}&P^\dagger\left(\left(\frac{Lh^d}{\log L}\right)^{1/2}\max_{k=1,\ldots,J_L}|C_L^\dagger(b)|>\Delta\right)\leq\\ &\qquad\leq\sum_{k=1}^{J_L}P^\dagger\left(\left|\sum_{l=1}^{L}\sum_{i=1}^{N}\left(T_{il}^\dagger\left(c_k\right)-E^\dagger T_{il}^\dagger\left(c_k\right)\right)\right|>\Delta NL\left(\frac{\log L}{Lh^d}\right)^{1/2}\right).\end{aligned} \tag{S.13}$$

By (S.2), $|T_{il}^\dagger(c_k)|\leq h^{-d}(\sup K)^d$ and

$$\left|T_{il}^\dagger\left(c_k\right)-E^\dagger T_{il}^\dagger\left(c_k\right)\right|\leq 2(\sup K)^d h^{-d}.$$

Further, by (S.6)-(S.9), there is a constant $0 < D_1 < \infty$ such that

$$\begin{aligned} Var^{\dagger}\left(T_{il}^{\dagger}\left(c_k\right)\right) &\leq D_1 h^{-2d} s_L \left(1 + o_p\left(1\right)\right) \\ &= D_1 h^{-d}\left(\log L / \left(Lh^d\right)\right)\left(1 + o_p\left(1\right)\right). \end{aligned}$$

We therefore can apply Bernstein's inequality (Pollard, 1984, page 193) to obtain

$$\begin{aligned} & P^{\dagger}\left(\left|\sum_{l=1}^{L}\sum_{i=1}^{N}\left(T_{il}^{\dagger}\left(c_k\right) - E^{\dagger}T_{il}^{\dagger}\left(c_k\right)\right)\right| > \Delta N L\left(\frac{\log L}{Lh^d}\right)^{1/2}\right) \\ & \leq 2\exp\left(-\frac{1}{2}\frac{\Delta^2 N^2 L^2 \frac{\log L}{Lh^d}}{NLD_1 h^{-d}\left(1 + o_p\left(1\right)\right)\frac{\log L}{Lh^d} + \left(2/3\right)\Delta N (\sup K)^d h^{-d} L\left(\frac{\log L}{Lh^d}\right)^{1/2}}\right) \\ & = 2\exp\left(-\frac{1}{2}\frac{\Delta^2 N\left(\log L\right)^{1/2}\left(Lh^d\right)^{1/2}}{D_1\left(\log L / \left(Lh^d\right)\right)^{1/2}\left(1 + o_p\left(1\right)\right) + \left(2/3\right)\Delta (\sup K)^d}\right) \\ & = 2\exp\left(-\frac{\Delta N}{\left(4/3\right)\left(\sup K\right)^d + o_p\left(1\right)}\left(\log L\right)^{1/2}\left(Lh^d\right)^{1/2}\right), \end{aligned} \tag{S.14}$$

where the equality in the last line is due to $Lh^d / \log L \to \infty$. The inequalities in (S.12)-(S.14) together with (S.3) imply that there is a constant $0 < D_2 < \infty$ such that

$$\begin{aligned} & P^{\dagger}\left(\left(\frac{Lh^d}{\log L}\right)^{1/2}\sup_{b \in I}\left|C_L^{\dagger}(b)\right| > \Delta\right) \\ & \leq 2J_L \exp\left(-\frac{\Delta N}{\left(4/3\right)\left(\sup K\right)^d + o_p\left(1\right)}\left(\log L\right)^{1/2}\left(Lh^d\right)^{1/2}\right) \\ & \leq D_2 s_L^{-1}\exp\left(-\frac{\Delta N}{\left(4/3\right)\left(\sup K\right)^d + o_p\left(1\right)}\left(\log L\right)^{1/2}\left(Lh^d\right)^{1/2}\right) \\ & \leq D_2 \exp\left(\log L\left(1 - \frac{\Delta N}{\left(4/3\right)\left(\sup K\right)^d + o_p\left(1\right)}\left(\frac{Lh^d}{\log L}\right)^{1/2}\right)\right) \\ & = o_p\left(1\right), \end{aligned}$$

where the equality in the last line is by $Lh^d / \log L \to \infty$. By a similar argument as in the proof of Lemma S.2,

$$\sup_{b \in I}\left|C_L^{\dagger}(b)\right| = o_p^{\dagger}\left(\frac{Lh^d}{\log L}\right)^{-1/2}. \tag{S.15}$$

The result of part (d) follows from (S.10), (S.11), and (S.15).

The proof of part (e) is similar to that of Lemma 3(e) in MSX. First, by similar arguments as in the proof of Lemma 3(e), one can show that $\underline{b}(N,x) \le \hat{q}^{*,\dagger}(\varepsilon|B,x) \le \hat{q}^{\dagger}(1-\varepsilon|n,x) \le \bar{b}(N,x)$ with probability $P^{\dagger}$ approaching one (in probability), and that uniformly over $\tau \in [\varepsilon, 1-\varepsilon]$,

$$\hat{G}^{*,\dagger}\left(\hat{q}^{\dagger}(\tau|N,x)|N,x\right) = \tau + O_p^{\dagger}\left(Lh^d\right)^{-1}$$

Next,

$$\begin{aligned}
&G^{*}\left(\hat{q}^{*,\dagger}(\tau|N,x)|N,x\right) - \hat{G}^{*,\dagger}\left(\hat{q}^{*,\dagger}(\tau|N,x)|N,x\right) \\
&= G^{*}\left(\hat{q}^{*,\dagger}(\tau|N,x)|N,x\right) - \tau + O_p^{\dagger}\left(Lh^d\right)^{-1} \\
&= G^{*}\left(\hat{q}^{*,\dagger}(\tau|N,x)|N,x\right) - G^{*}\left(q^{*}\left(\tau|N,x\right)|N,x\right) + O_p^{\dagger}\left(Lh^d\right)^{-1} \\
&= g^{*}\left(\tilde{q}^{*,\dagger}\left(\tau|N,x\right)|N,x\right)\left(\hat{q}^{*,\dagger}(\tau|N,x) - q^{*}\left(\tau|N,x\right)\right) + O_p^{\dagger}\left(Lh^d\right)^{-1},
\end{aligned}$$

where $\tilde{q}^{\dagger}$ denotes the mean value, or

$$\begin{aligned}
&\hat{q}^{*,\dagger}(\tau|N,x) - q^{*}\left(\tau|N,x\right) \\
&= \frac{G^{*}\left(\hat{q}^{*,\dagger}(\tau|N,x)|N,x\right) - \hat{G}^{*,\dagger}\left(\hat{q}^{*,\dagger}(\tau|N,x)|N,x\right)}{g^{*}\left(\tilde{q}^{*,\dagger}\left(\tau|N,x\right)|N,x\right)} + O_p^{\dagger}\left(Lh^d\right)^{-1}.
\end{aligned}$$

By part (d) of this lemma, Lemma 3(d) in MSX, and Lemma S.1(b),

$$\sup_{\tau\in[\varepsilon,1-\varepsilon]}\left|\hat{q}^{*,\dagger}(\tau|N,x) - q^{*}\left(\tau|N,x\right)\right| = O_p^{\dagger}\left(Lh^d\right)^{-1}. \tag{S.16}$$

As in the proof of Lemma S.1 and since $\hat{q}^{*}\left(\tau|N,x\right)$ is non-random under $P^{\dagger}$, for all $\epsilon > 0$ there is $\Delta_{\epsilon} > 0$ such that

$$\begin{aligned}
&P\left(P^{\dagger}\left(Lh^d\left|\hat{q}^{*}\left(\tau|N,x\right) - q^{*}\left(\tau|N,x\right)\right| > \Delta_{\epsilon}\right) > \epsilon\right) \\
&= P\left(1\left(Lh^d\left|\hat{q}^{*}\left(\tau|N,x\right) - q^{*}\left(\tau|N,x\right)\right| > \Delta_{\epsilon}\right) > \epsilon\right) \\
&= P\left(Lh^d\left|\hat{q}^{*}\left(\tau|N,x\right) - q^{*}\left(\tau|N,x\right)\right| > \Delta_{\epsilon}\right) \\
&< \epsilon,
\end{aligned} \tag{S.17}$$

where the inequality in the last line is by 3(e) in MSX. Furthermore, the last result holds uniformly in $\tau \in [\varepsilon, 1-\varepsilon]$. The result in part (e) of the lemma then follows by

(S.16) and (S.17).[1]

The result in part (f) is implied by Proposition 3.2 in Chen and Lo (1997). The proof of parts (g) and (h) is similar to that of Lemma 3(g) and (h) in MSX. ■

# References


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[1] Note that (S.17) establishes a trivial result that, if $\hat{\theta}_L = \theta + O_p(\lambda_L)$, then $\hat{\theta}_L = \theta + O_p^{\dagger}(\lambda_L)$ (recall that $\hat{\theta}_L$ is computed using the data in the original sample).